A Multiple Integral Explicit Evaluation Inspired by The Multi-WZ Method

Akalu Tefera[1]

Abstract

We give an identity which is conjectured and proved by using an implementation [3] in multi-WZ [5].

0. Introduction

There are relatively few known non-trivial evaluations of n-dimensional integrals, with *arbitrary* n. Celebrated examples are the Selberg and the Metha-Dyson integrals, as well as the Macdonald constant term ex-conjectures for the various infinite families of root systems. They are all very important. See [1] for a superb exposition of the various known proofs and of numerous intriguing applications.

At present, the (continuous version of the) WZ method[5] is capable of mechanically proving these identities only for a fixed n. In principle for *any* fixed n (even, say, $n = 100000$), but in practice only for $n \leq 5$. However, by interfacing a human to the computer-generated output, the human may discern a pattern, and generalize the computer-generated proofs for $n = 1, 2, 3, 4$ to an arbitrary n.

Using this strategy, Wilf and Zeilberger[5] gave a WZ-style proof of Selberg's integral evaluation. But just giving yet another proof of an already known identity, especially one that already had (at least) three beautiful proofs (Selberg's, Aomoto's, and Anderson's, see [1]), is not very exciting.

In this article we present a *new* multi-integral evaluation, that was *first* found using the author's implementation of the continuous multi-WZ method[3]. Both the conjecturing part, and the proving part, were done by a close human-machine collaboration. Our proof hence may be termed *computer-assisted* but not yet *computer-generated.*

[1]Department of Mathematics, Temple University, Philadelphia, PA 19104.
akalu@math.temple.edu

Now that the result is known and proved, it may be of interest to have a non-WZ proof, possibly by performing an appropriate change of variables, converting the multi-integral to a double integral. My advisor, Doron Zeilberger, is offering \$100 for such a proof, provided it does not exceed the length of the present proof.

1. Notation

$\mathbf{x} = (x_1, \dots, x_k)$, $\quad (y)_m = \prod_{i=0}^{m-1}(y+i)$,

$d\mathbf{x} = dx_1 \cdots dx_k$, $\quad e_1(\mathbf{x}) = \sum_{i=1}^{k} x_i$,

$\hat{\mathbf{x}}_i = (x_1, \dots, \hat{x}_i, \dots, x_k)$, $\quad e_2(\mathbf{x}) = \sum_{i<j} x_i x_j$,

$\Delta_n F(n, \mathbf{x}) = F(n+1, \mathbf{x}) - F(n, \mathbf{x})$

2. The Integral Evaluation

Theorem

$$\int_{[0,+\infty)^k} (e_2(\mathbf{x}))^m (e_1(\mathbf{x}))^n e^{-e_1(\mathbf{x})} d\mathbf{x} = \frac{m!(2m+n+k-1)!(k/2)_m}{(2m+k-1)!} \left(\frac{2(k-1)}{k}\right)^m T_k(m)$$

for all k in $\mathbb{N}$, and for all m, n in $\mathbf{Z}_{\geq 0}$, where,

$$T_k(m) - T_k(m-1) = \frac{(k(k-2))^m((k-1)/2)_m}{(k-1)^{2m}(k/2)_m} T_{k-1}(m)$$

for all $k \geq 2$, $T_1(m) = 0$, for all m in $\mathbf{Z}_{\geq 0}$, and $T_k(0) = 1$ for all $k \geq 2$.

3. Proof of the Integral Evaluation

If $k = 1$, then trivially, both sides of the integral equate to zero. Let $k > 1$ and $A_k(m, n)$ be the left side of the integral divided by

$$\frac{m!(2m+n+k-1)!(k/2)_m}{(2m+k-1)!} \left(\frac{2(k-1)}{k}\right)^m.$$

We want to show $A_k(m, n) = T_k(m)$, for all m, n in $\mathbf{Z}_{\geq 0}$. Let

$$F_k(m, n; \mathbf{x}) := \frac{(2m+k-1)!}{m!(2m+n+k-1)!(k/2)_m} \left(\frac{k}{2(k-1)}\right)^m (e_2(\mathbf{x}))^m (e_1(\mathbf{x}))^n e^{-e_1(\mathbf{x})}$$

We construct[2]

$$R(u;v_1,\dots,v_{k-1}) := \frac{u}{2m+n+k},$$

with the motive that

$$\text{(WZ 1)} \qquad \Delta_n F_k(m,n;\mathbf{x}) = -\sum_{i=1}^{k} D_{x_i}[R(x_i;\hat{\mathbf{x}}_i)F_k(m,n;\mathbf{x})].$$

Now, we verify (WZ 1),

$$\begin{aligned}
&\frac{F_k(m,n+1;\mathbf{x}) - F_k(m,n;\mathbf{x}) + \sum_{i=1}^{k} D_{x_i}[R(x_i;\hat{\mathbf{x}}_i)F_k(m,n;\mathbf{x})]}{F_k(m,n;\mathbf{x})} \\
&= \frac{F_k(m,n+1;\mathbf{x})}{F_k(m,n;\mathbf{x})} - 1 + \sum_{i=1}^{k} D_{x_i}[R(x_i;\hat{\mathbf{x}}_i)] + R(x_i;\hat{\mathbf{x}}_i)D_{x_i}\left[log\left(F_k(m,n;\mathbf{x})\right)\right] \\
&= \frac{e_1(\mathbf{x})}{2m+n+k} - 1 + \frac{k}{2m+n+k} + \\
&\quad \sum_{i=1}^{k}\left(\frac{n}{e_1(\mathbf{x})}\frac{x_i}{2m+n+k} + \frac{me_1(\hat{\mathbf{x}}_i)}{e_2(\mathbf{x})}\frac{x_i}{2m+n+k} - \frac{x_i}{2m+n+k}\right) \\
&= \frac{e_1(\mathbf{x})}{2m+n+k} - 1 + \frac{k}{2m+n+k} + \frac{n}{2m+n+k} + \frac{2m}{2m+n+k} - \frac{e_1(\mathbf{x})}{2m+n+k} \\
&= 0.
\end{aligned}$$

Hence, by integrating both sides of (WZ 1) w.r.t $x_1,\dots,x_k$ over $[0,\infty)^k$, we get

$$A_k(m,n+1) - A_k(m,n) \equiv 0.$$

To complete the proof we show $A_k(m,0) = T_k(m)$.

To this end, set $A_k(m) := A_k(m,0)$ and $F_k(m;\mathbf{x}) := F_k(m,0;\mathbf{x})$. Now, we construct[3],

$$R(u;v_1,\dots,v_{k-1}) := \frac{((k-1)(m+1) + e_1(v_1,\dots,v_{k-1}))u + e_2(v_1,\dots,v_{k-1})}{(k-1)(m+1)(2m+k)}$$

with the motive that

$$\text{(WZ 2)} \qquad F_k(m+1;\mathbf{x}) - F_k(m;\mathbf{x}) = -\sum_{i=1}^{k} D_{x_i}[R(x_i;\hat{\mathbf{x}}_i)F_k(m;\mathbf{x})].$$

[2]for specific k, the rational function R is obtained by using **SMint** [3] and the output is available from *http://www.math.temple.edu/∼akalu/maplepack/rational1.output*

[3] for specific k, the rational function R is obtained by using **SMint** [3] and the output is available from *http://www.math.temple.edu/∼akalu/maplepack/rational2.output*

Verification of (WZ 2):

$$
\begin{aligned}
&\frac{F_k(m+1;\mathbf{x}) - F_k(m;\mathbf{x}) + \sum_{i=1}^{k} D_{x_i}[R(x_i;\hat{\mathbf{x}}_i)F_k(m;\mathbf{x})]}{F_k(m;\mathbf{x})} \\
= \;& \frac{F_k(m+1;\mathbf{x})}{F_k(m;\mathbf{x})} - 1 + \sum_{i=1}^{k} D_{x_i}[R(x_i;\hat{\mathbf{x}}_i)] + \sum_{i=1}^{k} R(x_i;\hat{\mathbf{x}}_i) D_{x_i}\left[log\left(F_k(m;\mathbf{x})\right)\right] \\
= \;& \frac{ke_2(\mathbf{x})}{(m+1)(k-1)(2m+k)} - 1 + \sum_{i=1}^{k} \frac{(k-1)(m+1) + e_1(\hat{\mathbf{x}}_i)}{(m+1)(k-1)(2m+k)} + \\
& \sum_{i=1}^{k} \frac{(k-1)(m+1)x_i + e_2(\mathbf{x}))}{(m+1)(k-1)(2m+k)} \left(\frac{me_1(\hat{\mathbf{x}}_i)}{e_2(\mathbf{x})} - 1\right) \\
= \;& \frac{ke_2(\mathbf{x})}{(m+1)(k-1)(2m+k)} - 1 + \frac{k}{2m+k} + \frac{e_1(\mathbf{x})}{(m+1)(2m+k)} + \frac{2m}{2m+k} - \frac{e_1(\mathbf{x})}{2m+k} + \\
& \frac{me_1(\mathbf{x})}{(m+1)(2m+k)} - \frac{ke_2(\mathbf{x})}{(m+1)(k-1)(2m+k)} \\
= \;& 0.
\end{aligned}
$$

Hence, by integrating both sides of (WZ 2) w.r.t. $x_1, \ldots, x_k$ over $[0,\infty)^k$, we obtain,

$$
A_k(m+1) - A_k(m) = \frac{(k(k-2))^{m+1}((k-1)/2)_{m+1}}{(k-1)^{2(m+1)}(k/2)_{m+1}} A_{k-1}(m+1),
$$

and noting that $A_k(0) = 1$, $A_1(m) = 0$, it follows that $A_k(m) = T_k(m)$, for all m in $\mathbf{Z}_{\geq 0}$. Consequently, $A_k(m,n) = T_k(m)$ for all m, n in $\mathbf{Z}_{\geq 0}$. □

By unfolding the recurrence equation for $T_k(m)$, we obtain the following identity.

Corollary

$$
\int_{[0,+\infty)^k} (e_2(\mathbf{x}))^m (e_1(\mathbf{x}))^n e^{-e_1(\mathbf{x})} d\mathbf{x} = \frac{m!(2m+n+k-1)!(k/2)_m}{(2m+k-1)!} \left(\frac{2(k-1)}{k}\right)^m
$$
$$
\left(1 + \sum_{r=1}^{k-2} \sum_{1 \leq s_r \leq \cdots \leq s_1 \leq m} \prod_{i=1}^{r} \frac{((k-i)^2-1)^{s_i}((k-i)/2)_{s_i}}{(k-i)^{2s_i}((k-i+1)/2)_{s_i}}\right)
$$

4. Remarks

1. From the computational point of view, the recurrence form of the integral is *nicer* than its indefinite summation form (the above corollary), for the former requires $O(mk)$ calculations, whereas the latter requires $O(m^k)$ calculations. However, in both forms the evaluation of the right side of the integral is much faster (for specific m, n, and k) than the direct evalution of the left side of our intergal. Hence both forms are indeed complete *answers* in the sense of Wilf[4].

2. The present paper is an example of what Doron Zeilberger[6] calls *WZ Theory, Chapter 1 1/2.* Even though, at present, our proof, for general n, was human-generated, it looks almost computer-generated. It seems that by using John Stembridge's[2] Maple package for symmetric functions, SF, or an extension of it, it should be possible to write a new version of `SMint` that should work for *symbolic*, i.e. arbitrary, n, thereby fulfilling the hope raised in [6].

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